B-math 2nd year Final Exam
Subject : Analysis

Time : 3.00 hours Max.Marks 75.

1. Let S be the intersection of the sphere $x^2 + y^2 + z^2 \leq a^2$ and the plane $x + y + z = 0$. Let C be the boundary of S.
a) Show using a suitable parametrization that

$$\int_C ydx + zdy + xdz = \sqrt{3}\pi a^2.$$

b) Verify your result using Stokes theorem. (10 + 10)

2. a) Let $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a continuous function. Show that

$$\int_0^\infty \int_0^\infty f(u, v)\ dudv = \int_0^\infty \int_0^1 f(s - st, st)\ dsdt$$

where the LHS is defined as the limit as n goes to infinity of $\int_0^n \int_0^n f(u, v)\ dudv$ and this limit is taken to be finite. A similar definition holds for the RHS.
b) Evaluate the integral

$$\int_0^1 (1 - t)^{x-1} t^{y-1}\ dt$$

for x, y positive integers by a suitable choice of $f(u, v)$ in a). Hint : Note that f has to be in product form. (10 + 5)

3. Let $V(t)$ be a solid sphere of radius $t > 0$ with center at a point $a \in I\!R^3$ and $S(t) := \partial V(t)$, the boundary of $V(t)$. Let $\vec{F}$ be a vector field that is continuously differentiable on $V(t)$. Then if $|V(t)| := Volume\ of\ V(t)$ and $\vec{n}$ is the outward unit vector normal to $S(t)$ show that

$$\nabla \cdot \vec{F}(a) = \lim_{t \to 0} \frac{1}{|V(t)|} \int_{S(t)} \vec{F} \cdot \vec{n}\ dS.$$

(10)

4. Consider the system of equations

$$\begin{aligned} x^2 - y^2 + u^2 + v^2 &= 1 \\ x^2 + y^2 - u^2 - v^2 &= 2 \end{aligned}$$

a) Show that there exists infinitely many points $(x, y, u, v) \in I\!R^4$ which satisfy the above equation.
b) For points as in a) can you obtain u and v as functions of x, y such that $(x, y, u(x, y), v(x, y))$ satisfy the above equations on a suitable open set ? Justify your answer. (5+5)

5. a) Let $f_n(x) := \frac{1}{nx+1}, 0 < x < 1$. Show that f_n does not converge uniformly on $(0, 1)$.
b) Let ϕ be defined on $I\!R$ and satisfy $\phi(x) = |x|, -1 \leq x \leq 1$ and $\phi(x+2) = \phi(x), x \in I\!R$. Show that the series $\sum\limits_{n=0}^{\infty} (\frac{3}{4})^n \phi(4^n x)$ converges for each $x \in I\!R$ and defines a continuous function. (7+8)

6. Let g and h be smooth functions on $A := \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. Show that

$$\int_A (g\Delta h + \nabla g \cdot \nabla h)\, dV = \int_{\partial A} g \nabla h \cdot \vec{n}\, dS$$

where $\vec{n}$ is the unit outward normal to the boundary ∂A, and $\Delta h(x) := \sum_{i=1}^{3} \partial_i^2 h(x)$ for x in a neighborhood of V. Hint : Consider the vector field $\vec{F}(x) := g(x)\nabla h(x)$. (15)